Gaussian Parts of Compact Quantum Groups

Uwe Franz

University of Franche-Comté, Besançon

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Goal of this talk

- On Lie groups, Brownian motions are defined as stochastic processes with independent and stationay increments (i.e., Lévy processes) and continuous paths.
- Taking Gaussianity, as defined by Schürmann, see [Sch93], as a substitute for path continuity, we define an analog of the connected component of the identity of a compact quantum groups as its smallest subgroup that supports all its Gaussian Lévy processes.
- We also consider a second stronger notion of connectedness.
- At the end of the talk we will look at ongoing work on extending these notions of braided compact quantum groups.

Joint work with Amaury Freslon & Adam Skalski, Biswarup Das & Anna Kula & Adam Skalski, Sutanu Roy.

Programme

1. Introduction
2. Gaussian Lévy processes
3. Gaussian part and strongly connnected component of the identity
4. Generalization to braided CQG and braided Lévy processes (work in progress)
5. Open problems

Lévy processes, etc.

Let $B = \mathrm{Pol}(\mathbb{G})$ be the involutive Hopf algebra (CQG algebra) of a compact quantum group $\mathbb{G}$.

We have bijections between the following objects:

- $\big(j_{st} : B \to (A, \Phi)\big)_{0 \leqslant s \leqslant t < \infty}$ a Lévy process on B (or $\mathbb{G}$) over some quantum probability space (A, Φ)
- $(\varphi_t)_{t \geqslant 0}$ with $\varphi_{t-s} = \Phi \circ j_{st}$ a convolution semigroup of states on B
- $L : B \to \mathbb{C}$ with $L = \frac{d}{dt}\Big|_{t=0} \varphi_t$ a generating functional, i.e., a hermitian linear functional s.t. $L(1) = 0$ and L is positive on $K_1 = \ker(\varepsilon)$.

Lévy processes, etc., cont'd

- $(\pi : B \to B(H), \eta : B \to H, L : B \to \mathbb{C})$ a Schürmann triple over some Hilbert space H. I.e., a triple of linear maps s.t.

$$B \ni a \mapsto \beta(a) = \begin{pmatrix} \varepsilon(a) & \langle \eta(a^*)| & L(a) \\ 0 & \pi(a) & |\eta(a)\rangle \\ 0 & 0 & \varepsilon(a) \end{pmatrix} \in \mathrm{Lin}(\mathbb{C} \oplus H \oplus \mathbb{C})$$

is a unital *-homomorphism, if we equip $\mathbb{C} \oplus H \oplus \mathbb{C}$ with the (non-positive) inner product

$$\langle x \oplus v \oplus y, x' \oplus v' \oplus y' \rangle = \overline{x} y' + \overline{y} x' + \langle v, v' \rangle_H.$$

Remark

The triple (π, η, L) is uniquely determined up to unitary equivalence, if we require $\eta(B)$ to be dense in H, as we shall do from now on.

Schürmann's definition of Gaussianity

For $n \geqslant 1$ let

$$K_n = \text{span}\{a_1 \cdots a_n : a_1, \ldots, a_n \in \ker(\varepsilon)\}, \qquad K_\infty = \bigcap_{n \geqslant 1} K_n.$$

Definition

A Lévy process (j_{st}), its states (φ_t), its generating functional L, and its Schürmann triple (π, η, L) are called Gaussian, if one, and therefore all the following conditions hold.

- $L|_{K_3} = 0$
- $L(abc) =$
  $L(ab)\varepsilon(c) + L(ac)\varepsilon(b) + L(bc)\varepsilon(a) - L(a)\varepsilon(bc) - L(b)\varepsilon(ac) - L(c)\varepsilon(ab)$
- $\eta|_{K_2} = 0$
- $\eta(ab) = \varepsilon(a)\eta(b) + \eta(a)\varepsilon(b)$
- $\pi|_{K_1} = 0$
- $\pi(\cdot) = \varepsilon(\cdot)\text{Id}_H$

Let $\mathbb{H}$ be a compact quantum subgroup of a compact quantum group $\mathbb{G}$, i.e., suppose there exists a surjective unital *-homomorphism

$$q_{\mathbb{H}} : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{H})$$

satisfying $\Delta_{\mathbb{H}} \circ q_{\mathbb{H}} = (q_{\mathbb{H}} \otimes q_{\mathbb{H}}) \circ \Delta_{\mathbb{G}}$.

Definition

We say that a generating functional $L : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ factors through $\mathbb{H}$ (or "lives on $\mathbb{H}$"), if there exists a linear functional $L_{\mathbb{H}} : \mathrm{Pol}(\mathbb{H}) \to \mathbb{C}$ such that

$$L = L_{\mathbb{H}} \circ q_{\mathbb{H}}$$

In that case $L_{\mathbb{H}}$ is automatically also a generating functional.

The Gaussian part, cont'd

Definition

The Gaussian part $\mathrm{Gauss}(\mathbb{G})$ of a compact quantum group $\mathbb{G}$ is defined as the intersection of all quantum subgroups of $\mathbb{G}$ through which all Gaussian generating functionals factor.

I.e., $\mathrm{Pol}\big(\mathrm{Gauss}(\mathbb{G})\big)$ is the quotient of $\mathrm{Pol}(\mathbb{G})$ by the larged Hopf *-ideal contained in

$$\bigcap_{L \text{ Gauss gen funct}} \ker(L) \supseteq K_3.$$

The strongly connected component of the identity

Let $\mathbb{G}$ be a compact quantum group.

Lemma

$K_\infty = \bigcap_{n \geqslant 1} K_n$ is a Hopf *-ideal.

Definition

We define the strongly connected component of the identity of $\mathbb{G}$ as the compact quantum group $\mathbb{G}^{00}$ with CQG algebra

$$\mathrm{Pol}(\mathbb{G}^{00}) = \mathrm{Pol}(\mathbb{G})/K_\infty.$$

The strongly connected component of the identity, cont'd

Remarks

1. Since
$$K_\infty \subseteq K_3 \subseteq \bigcap_{L \text{ Gauss gen funct}} \ker(L),$$
we have $\mathrm{Gauss}(\mathbb{G}) \subseteq \mathbb{G}^{00}$.
2. We also have $\mathbb{G}^{00} \subseteq \mathbb{G}^{0}$, where $\mathbb{G}^{0}$ denotes the connected component of the identity of $\mathbb{G}$ defined by Cirio, D'Andrea, Pinzari, and Rossi (2014).

No component group

Remark

In general the subgroups $\mathrm{Gauss}(\mathbb{G})$ and $\mathbb{G}^{00}$ are not normal, so that there is no analog of the component group.

The Commutative Case

Theorem

If G is a classical compact group, then its Gaussian part coincided with the (strongly) connected component of its identity,

$$\mathrm{Gauss}(G) = G^{00} = G^{0}.$$

The Cocommutative Case

Theorem

If Γ is a finitely generated discrete group, then

$$\mathrm{Gauss}(\widehat{\Gamma}) = \Gamma/\widehat{\sqrt{\gamma_3(\Gamma)}}.$$

In particular, $\widehat{\Gamma}$ is Gaussian, iff it is torsion-free nilpotent of class 2 (i.e., it is torsion free and all the commutators are central).

Furthermore, $\widehat{\Gamma}$ is strongly connected iff Γ is 'residually torsion-free nilpotent' (i.e., for any $\gamma \in \Gamma$, $\gamma \neq e$, there exists a normal subgroup $N \subseteq \Gamma$ s.t. $\gamma \notin N$ and Γ/N is torsion-free).

q-deformed semisimple compact Lie groups

Theorem

For G be a simply connected semisimple compact Lie group and $0 < q < 1$, we have

$$\mathrm{Gauss}(G_q) = (G_q)^{00} = \mathrm{Kac}(G_q) = \mathbb{T},$$

where $\mathbb{T} \subseteq G$ is the maximal torus.

Tomatsu (2007) showed $\mathrm{Kac}(G_q) = \mathbb{T}$, and the rest follows since the torus is connected and classical.

Main Theorem

At the times of Adam's talk in this seminar (i.e., October 2021), we knew that $\mathrm{Gauss}(\mathbb{G}) \subseteq \mathbb{G}^{00}$ and $\mathrm{Gauss}(\mathbb{G}) \subseteq \mathrm{Kac}(\mathbb{G})$.
In the published paper [FFS23] we proved the following stronger result.

Theorem

The Gaussian part of $\mathbb{G}$ is contained in the strongly connected component of its identity, which is contained in its Kac part:

$$\mathrm{Gauss}(\mathbb{G}) \subseteq \mathbb{G}^{00} \subseteq \mathrm{Kac}(\mathbb{G}).$$

The proof of the inclusion $\mathbb{G}^{00} \subseteq \mathrm{Kac}(\mathbb{G})$ uses the unitarity relations for u and $Q\overline{u}Q^{-1}$, with u some unitary irrep, see Proposition 4.11 in [FFS23].

Braided Hopf algebras and compact quantum groups

Recall that a braided monoidal category $(\mathcal{C}, \Psi)$ is a tensor category $\mathcal{C}$ equiped with a braiding Ψ, i.e., a family of natural isomorphisms $\Psi_{A,B} : A \otimes B \to B \otimes A$ satifying in particular the hexagon identities. It is called a symmetry if $\Psi_{B,A} \circ \Psi_{A,B} = \mathrm{Id}_{A \otimes B}$.

Very roughly, braided bialgebras, braided Hopf algebras, and braided compact quantum groups are objects in braided monoidal category satisfying the usual axioms, but with the flip map $\tau_{B,B} : B \otimes B \to B \otimes B$, $\tau(a \otimes b) = b \otimes a$ replaced by the braiding $\Psi_{B,B}$.

In [FSV23] we generalized Schürmann's "symmetrization" of Lévy processes on "twisted involutive bialgebras" (twisted by a group action) to braided involutive bialgebras, i.e., involutive bialgebras in a braided monoidal category (e.g., the category of Yetter-Drinfeld modules of some involutive Hopf algebra).

Braided independence and braided Lévy processes

Braided Lévy processes on a braided involutive bialgebra can be defined similarly to usual (symmetric) Lévy processes, but braided independence requires the increments to satisfy commutation relations determined by the braiding Ψ,

$$m_A \circ (j_{s't'} \otimes j_{st}) = m_A \circ (j_{st} \otimes j_{s't'}) \circ \Psi_{B,B}$$

for $0 \leqslant s \leqslant t \leqslant s' \leqslant t'$.

Lévy processes on a braided involutive bialgebra are again classified by their generating functionals, which now have to be furthermore Ψ-invariant, i.e., satisfy

$$(\mathrm{id}_X \otimes L) \circ \Psi_{X,B} = L \otimes \mathrm{id}_X : B \otimes X \to \mathbb{C} \otimes X \cong X.$$

for any object X.

Gaussianity can be defined as before.

Can we generalize the Main Theorem to braided compact quantum groups?

In joint work with Sutanu Roy, we showed that the Gaussian generating functionals on all known examples of $\mathbb{T}$-braided CQG factor through a trivially braided quantum subgroup of Kac type.

Since the antipode in braided CQG satisfies the same identity as in trivially braided CQG (i.e., the special case $\Psi = \tau$), we have the following (weaker) general result.

Theorem

Let B be a braided involutive Hopf algebra. If the ideal

$$\langle \mathrm{Im}(S^2 - \mathrm{id}) \rangle$$

generated by the image of $S^2 - \mathrm{id}$ is also a coideal, then all Gaussian generating functionals on B factor through its "Kac quotient"

$$B_{\mathrm{Kac}} = B / \langle \mathrm{Im}(S^2 - \mathrm{id}) \rangle.$$

An example of a non-trivially braided Gaussian process

Let $q \in \mathbb{C}\backslash\{0\}$. Consider $B = \mathbb{C}\langle x, x^*\rangle$ the unital free algebra with one generator, with the braiding $\Psi : B \otimes B \to B \otimes B$ determined by

$$\begin{aligned} \Psi(x \otimes x) &= qx \otimes x, & \Psi(x^* \otimes x) &= q^{-1}x \otimes x^*, \\ \Psi(x \otimes x^*) &= \overline{q}x^* \otimes x, & \Psi(x^* \otimes x^*) &= \overline{q}^{-1}x^* \otimes x^*, \end{aligned}$$

This algebra can be turned into a braided involutive Hopf algebra s.t. x is primitive, i.e. $\Delta(x) = x \otimes 1 + 1 \otimes x$.

An example of a non-trivially braided Gaussian process, cont'd

Use words in x and x^* as a basis of B and define a linear functional $L : B \to \mathbb{C}$ by

$$L(w) = \begin{cases} \alpha & \text{if } w = x^*x, \\ \beta & \text{if } w = xx^*, \\ 0 & \text{else}, \end{cases}$$

with $\alpha, \beta \geqslant 0$.
This is a Gaussian generating functional, and the associated Gaussian Lévy process has been studied by Michael Schürmann, see [Sch91], [Sch93].

The increments $j_{st}(x)$ and $j_{s't'}(x)$ with $0 \leqslant s \leqslant t \leqslant s' \leqslant t'$ q-commute. The distribution of $j_{st}(x + x')$ (= the quantum Azéma process) depends on q, i.e., this process "sees" the braiding.

But $\langle \mathrm{Im}(S^2 - \mathrm{id}) \rangle \subseteq B$ is not a coideal. And B is not a braided CQG algebra.

References

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Open Problems

- What are the Gaussian parts of the universal unitary and orthogonal compact quantu, groups,

$$\mathrm{Gauss}(U_N^+) = ? \qquad \mathrm{Gauss}(O_N^+) = ?$$

- We need a better understanding of braided compact quantum groups. Do braided CQG have a “maximal trivially braided subgroup” or a “maximal Kac-type subgroup”?
- Does our main theorem generalize to braided CQG?

Thank you!